The best proof of Cousin's lemma

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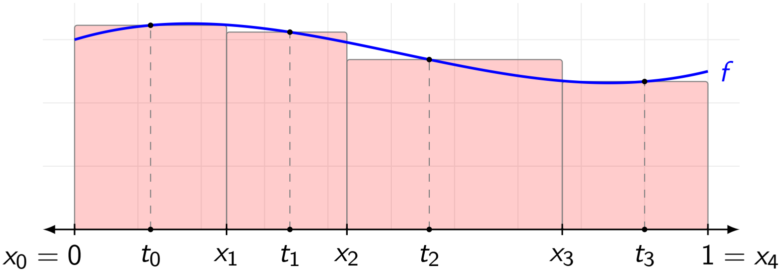
Reverse mathematics

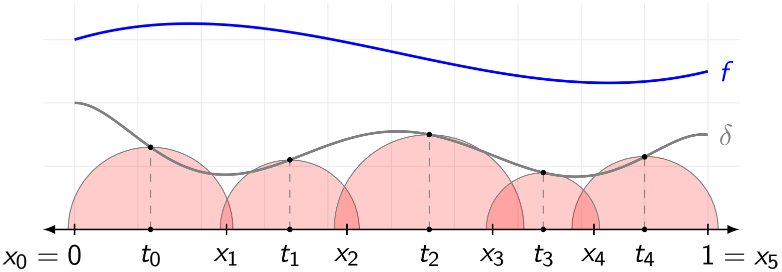
- For us, best proof $=$ least assumptions (axioms)
- Find weakest axiom system $\mathcal{S}$ which can prove a theorem φ
- Almost all theorems are equivalent to one of five systems
- In order of increasing strength:
  - RCA_0: computable mathematics
  - WKL_0: compactness
  - ACA_0: arbitrary quantification over $\mathbb{N}$
  - ATR_0: ordinals
  - Π^1_1-CA_0: quantification over $\mathbb{R}$ or $\mathcal{P}(\mathbb{N})$

Riemann integration



- Partition $[0, 1]$, pick a tag point in each subinterval
- Want these approximations to converge as $\Delta x \to 0$

Gauge integration



- Gauge: positive-valued function $\delta\colon [0,1] \to \mathbb{R}^+$
- $P = \langle x_i, t_i \rangle$ is δ-fine if $(x_i, x_{i+1}) \subseteq B\big(t_i, \delta(t_i)\big)$ for all i

Cousin's lemma

Cousin's lemma

Every gauge $\delta : [0,1] \to \mathbb{R}^+$ has a δ-fine partition.

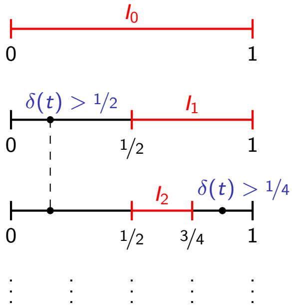
Proof.

Ask: is there any point $t \in [0,1]$ such that $\delta(t) > 1$?

Yes: then $P = \langle 0, t, 1 \rangle$ is δ-fine.

No: split $[0,1]$ in half, and see if either half has $\delta(t) > 1/2$, etc.

Must terminate: else we get $I_0 \supsetneq I_1 \supsetneq \cdots$. Pick $r \in \bigcap I_n$, then $\delta(r) = 0$; contradiction! $\square$

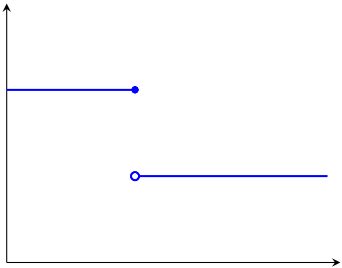


Question

Is this the *best* proof?

Our contributions

- This proof uses quantification over $\mathbb{R} \implies \Pi^1_1\text{-CA}_0$!
- Can we do better? For *continuous* gauges: yes!
- Theorem: CL for continuous functions equivalent to WKL_0.
- Baire 1 = pointwise limit of continuous functions
- $\text{ACA}_0 \leq \text{CL}_{\text{B1}} \leq \Pi^1_1\text{-CA}_0$
- Baire 2 = ptwise limit of B1
- $\text{ATR}_0 \leq \text{CL}_{\text{B2}} \leq \Pi^1_1\text{-CA}_0$



References

- Jordan Mitchell Barrett. *The reverse mathematics of Cousin's lemma.* Honours thesis, VUW, 2020. URL: `jmbarrett.nz`

For more reading on reverse mathematics:

- Stephen G. Simpson. *Subsystems of Second Order Arithmetic.* 2nd ed. Perspectives in Logic, Cambridge University Press, Cambridge, 2009.
- John Stillwell. *Reverse Mathematics: Proofs from the Inside Out.* Princeton University Press, Princeton, 2018.