The reverse mathematics of Cousin's lemma

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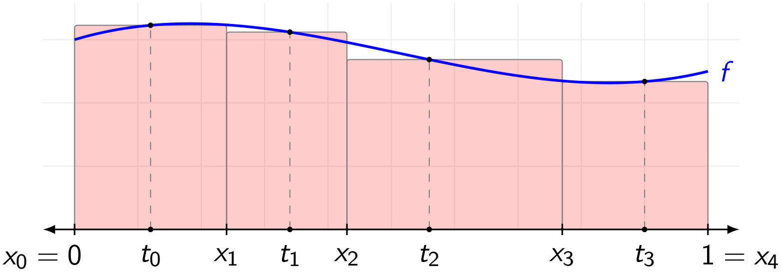
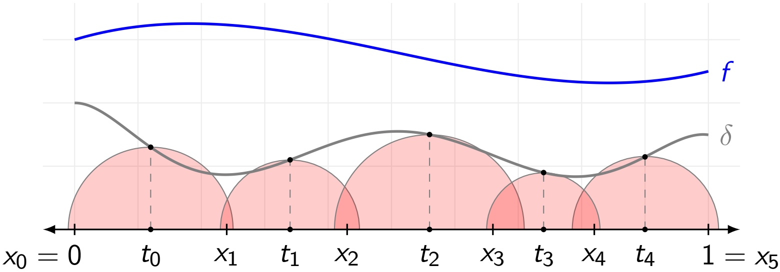
Riemann integration



- Let $f\colon [0,1] \to \mathbb{R}$ be a continuous function.
- A tagged partition of $[0,1]$ is a finite increasing sequence $P = \langle 0 = x_0 < t_0 < x_1 < t_1 < \cdots < t_{n-1} < x_n = 1 \rangle$
- Riemann sum: $\mathrm{RS}(f, P) := \sum_P f(t_i)[x_{i+1} - x_i]$
- Riemann integral: $\int_{\mathrm{R}} f = M$ if for all $\varepsilon > 0$, $\exists\, \delta > 0$ such that $|\mathrm{RS}(f, P) - M| < \varepsilon$ whenever blocks of P have size $< \delta$

Gauge integration



- Gauge: positive-valued function $\delta\colon [0,1] \to \mathbb{R}^+$
- $P = \langle x_i, t_i \rangle$ is δ-fine if $(x_i, x_{i+1}) \subseteq B\big(t_i, \delta(t_i)\big)$ for all i
- Gauge integral: $\int_G f = M$ if for all $\varepsilon > 0$, there is gauge δ such that $|\mathrm{RS}(f, P) - M| < \varepsilon$ whenever P is δ-fine
- If we only consider constant gauges $\implies$ Riemann integral
- ...what if there are *no* δ-fine partitions?

Cousin's lemma

Cousin's lemma

Every gauge $\delta : [0,1] \to \mathbb{R}^+$ has a δ-fine partition.

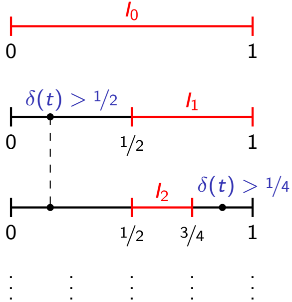
Proof.

Ask: is there any point $t \in [0,1]$ such that $\delta(t) > 1$?

Yes: then $P = \langle 0, t, 1 \rangle$ is δ-fine.

No: split $[0,1]$ in half, and see if either half has $\delta(t) > 1/2$, etc.

Must terminate: else we get $I_0 \supsetneq I_1 \supsetneq \cdots$. Pick $r \in \bigcap I_n$, then $\delta(r) = 0$; contradiction! $\square$



Question

Is this the *best* proof? What does that even mean?

Reverse mathematics

- What makes the best proof? Elegance, simplicity, clarity, ...
- Reverse mathematics: best = least assumptions (axioms)

"When the theorem is proved from the right axioms, the axioms can be proved from the theorem." —*Harvey Friedman*

- Given theorem φ, weak axiom system $\mathcal{S}$: try to prove $\mathcal{S} \vdash \varphi$
- If we can, then try get a reversal of φ: a proof $\mathcal{B} + \varphi \vdash \mathcal{S}$
  - Need base system $\mathcal{B}$: φ generally can't prove basic axioms
  - Shows proof $\mathcal{S} \vdash \varphi$ is *optimal*: can't use weaker axioms
- Why? Shows how (non)constructive a theorem is
  - How complex is the solution relative to the problem?

Second-order arithmetic

- Second-order arithmetic: numbers $n, m, k, \ldots$, sets $A, B, C, \ldots$
- Standard arithmetic operations $+$, $\cdot$, relations $=$, $<$, $\in$
- Have *number quantifiers* $\forall x$, $\exists x$, and *set quantifiers* $\forall X$, $\exists X$
- Subsystems of SOA comprise three main types of axioms:
  - Basic axioms: basic facts of arithmetic. Similar to PA
  - Induction axioms: say induction is valid over a formula φ

$$\left[\varphi(0) \wedge \forall n \left(\varphi(n) \rightarrow \varphi(n+1)\right)\right] \rightarrow \forall n \; \varphi(n)$$

  - Comprehension axioms: assert the existence of certain sets

$$\exists X \; \forall n \left[n \in X \leftrightarrow \varphi(n)\right]$$

Subsystems of second-order arithmetic

- In order of increasing strength:
- RCA_0: basic axioms + induction for Σ^0_1 formulae[a]+ comprehension for Δ^0_1 (computable) sets
  - Theorems of $\mathrm{RCA}_0 \approx$ theorems of computable mathematics
  - Base system: generally use $\mathcal{B} = \mathrm{RCA}_0$ in reversals
- WKL_0: RCA_0 + weak Kőnig's lemma (every infinite binary tree has an infinite branch)
- Arithmetical formula: no set quantifiers
- ACA_0: basic axioms + arithmetical comprehension/induction
- Π^1_1 formula: $\forall X\ \theta(X)$ for θ arithmetical
- Π^1_1-CA_0: basic axioms + Π^1_1 comprehension and induction

[a] $\exists x\, \theta(x)$, where $\theta(x)$ contains only bounded number quantifiers.

The reverse math zoo

Π^1_1-CA_0	Cantor–Bendixson; MBS lemma	
ACA_0	$\mathbb{R}$ complete; Maximal ideal; RT^k, $k \geq 3$; Bolzano–Weierstrass; V basis	
WKL_0	EVT; Heine–Borel; Prime ideal; Hahn–Banach; Gödel's compl.	RT^2_2
RCA_0	IVT; BCT; Soundness; $M = (E, \mathcal{I})$ basis	

Remember this proof?

Cousin's lemma

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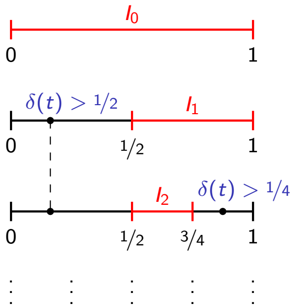


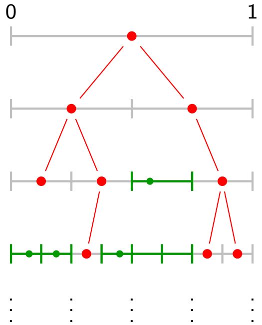
Question

Is this the *best* proof?

Formalising the proof in SOA

- Formally: build a binary tree T
- Nodes on level n are dyadic intervals $J = [\,{}^{i}/_{2^n},\ {}^{i+1}/_{2^n}\,]$
- Children of J are its two halves
- Put J in T if ancestors are, and

$$\underbrace{\forall r \in J,\ \delta(r) \leq |J|}_{\Pi^1_1 \text{ formula}}$$



- T must be finite, or WKL gives path $\implies \perp$ as before. $\square$
- Real numbers $\cong$ subsets of $\mathbb{N}$, so this proof needs Π^1_1-CA_0!
- Is this *really* the best we can do?

Our contributions

- For continuous functions, can do *much* better than Π^1_1-CA_0
  - Enough to consider midpoints, use sequential continuity

Theorem (Barrett, Downey, Greenberg; RCA_0)
Cousin's lemma for continuous functions is equivalent to WKL_0.

- Baire 0 = continuous, Baire $n+1$ = ptwise limit of Baire n
- Is Π^1_1-CA_0 the best we can do for Baire n functions? Maybe...

Theorem (Barrett, Downey, Greenberg; RCA_0)
Cousin's lemma for Baire 1 functions implies ACA_0.

Theorem (Barrett, Downey, Greenberg; RCA_0)
ACA_0 does **not** imply Cousin's lemma for Baire 2 functions.